Strange nonchaotic attractors in noise driven systems

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Abstract

Strange nonchaotic attractors (SNAs) in noise driven systems are investigated. Before the transition to chaos, due to the effect of noise, a typical trajectory will wander between the periodic attractor and its nearby chaotic saddle in an intermittent way, forms a strange attractor gradually. The existence of SNAs is confirmed by simulation results of various critera both in map and continuous systems. Dimension transition is found and intermittent behavior is studied by peoperties of local Lyapunov exponent. The universality and generalization of this kind of SNAs are discussed and common features are concluded.

PACS numbers: 05.45.-a, 05.40.-a

Since the pioneer work of Grebogi *et al.* in 1984 [1], strange nonchaotic attractors (SNAs) have been an important topic in nonlinear dynamics and studied widely both in theory and experiment [2, 3]. Here the word "strange" refers to the complicated geometry of the attractor, and the word "nonchaotic" refers to the no sensitive dependence on initial conditions. Therefore SNAs are denoted by systems which own fractal structure but with nonpositive Lyapunov exponent. SNAs not only play an important role in our study of transitions to chaos, but also a general phenomenon in physically relevant situations and related potential applications are also expected [4, 5, 6]. While the existence of SNAs is firmly established, a question that remains interesting is the mechanisms and routes through which SNAs are created. During the past years, five main mechanisms or scenarios for the creation of SNAs have been advanced: the birth of SNAs through collision between a period-doubled torus and its unstable parent [7]; the collision between a stable torus and an unstable one at a dense set of points [8]; the fractalization of a torus that the increasing wrinkling of tori leads to the appearance of SNAs without any interaction with a nearby unstable periodic orbit [9]; the loss of transverse stability of a torus [10]; the appearance of SNAs through type-I intermittency or type-III intermittency [11].

With our understanding, there are two features which are common for former studied SNAs: first, SNAs are found typically in qusiperiodically driven systems; second, the routes and mechanisms mentioned above are not general in the sense that they are model dependent, namely, has not a universal way which can be used to create SNAs in most models. Qusiperiod stands a middle position between period and random in that, on one hand, its spectrum is discrete which is similar to the periodic signal, and on the other hand, qusiperiodic signal never repeats itself in time series just like the random signal. Now we know that period can't induce SNAs, but an interesting question is whether SNAs can be induced by noise? if this is the situation, how does it works? this is still an open question which has never been studied before. In comparison with the qusiperiodic forcing, noise is more general and almost unavoidable in real systems, thus the study of noise induced SNAs not only will help our further understanding of some basic problems in nonlinear dynamics, but has a direct connection with experiments and applications as well. Meanwhile, we noticed that in studies of noise induced synchronization and noise induced transitions to chaos [12, 13], the trajectory often show some kind of intermittency and dimension has a dramatic change at some critical noise intensity, we also wish the study of SNAs in noise driven systems can

characterize some properties of these processes and explore some underlying mechanisms behind. Moreover, how to construct a SNA in a general chaotic system is still a challenge and an open question [14], by this study, we also wish to find a more general route for SNAs creation.

Our motivation comes from the recent works of Liu and Lai [13], where the transition to chaos in noise driven systems is investigated. One important result in their work is that noise can induce unstable dimension variability in a general sense. When noise intensity D is below the critical value D_c, a random initial condition leads to a trajectory confined in the vicinity of the periodic attractor except few transient chaos initially. When $D > D_c$, dramatic change appears where a typical trajectory doesn't be confined within the vicinity of the periodic attractor, it will wander between the periodic attractor and its nearby chaotic saddle, by this way a connection of unstable dimensions between two distinct sets is established. By studying the asymptotic behavior of the largest Lyapunov exponent λ, a power law relation is found between λ and D for $D \gtrsim D_c$, this relation is also valid for D versus the unstable dimension variability and the frequency of visit to chaotic saddle [13]. Things we interest here is, as the noise intensity exceeds D_c, although λ increases linearly, its still possible that λ is nonpositive within some range in the sense of asymptotic behavior, at the same time, a typical trajectory will wander randomly between two distinct dimension sets, the period state of $d = 0$ and the chaotic saddle of $d > 0$ (d represents the dimension), constructs a fractal dimension gradually. By definiting D_0 the critical noise intensity where λ exceeds 0, SNAs are expected to appear within range $D_c < D < D_0$. The main purpose of this work is just to analyze and characterize SNAs in this range, and try to find a more general way for SNAs creation.

As the first example, we study SNAs in noise driven logistic map with function

$$x_{n+1} = ax_n(1 - x_n) + D\xi_n, \tag{1}$$

where D represents the noise intensity and ξ_n is the Gaussian random variable of zero mean and unit variance. For $a = 3.8008$ and $D = 0$, system stays in a period 8 window with the largest Lyapunov exponent $\lambda_p \approx -0.127 < 0$. As D exceeds the critical value $D_c \approx 10^{-5.08}$, $\lambda > \lambda_p$ and increases with a power law scaling, $\lambda - \lambda_p \sim (D - D_c)^{-\alpha}$, with α a system dependent parameter. Further increasing D, when $D > D_0 \approx 10^{-4.95}$, the largest Lyapunov exponent changes to positive and system becames chaos. The process

that λ changes with D is plotted in Fig. 1(a), the range $D_c < D < D_0$ where $\lambda_p < \lambda < 0$ is shown. As we have analyzed above, in this range, due to the effect of noise, a typical trajectory will stays partially in the periodic attractor and partially in the chaotic saddle, we present this intermittent behavior in Fig. 1(b) by plotting variable x_n versus iteration time n for a random chosen initial condition. It can be found that this time the trajectory doesn't be confined within the vicinity of periodic attractor, but in some iteration intervals, its trajectory cruises the whole phase space with range $(0, 1)$, this made the dimension no longer zero, but some value between 0, the dimension of periodic attractor, and 1, the dimension of full developed chaotic attractor. Here we employ the information dimesion, $d = \lim_{\varepsilon \to 0} \frac{I}{\ln \varepsilon} = \lim_{\varepsilon \to 0} [\sum_{i=1,N} \mu_i \ln \mu_i] / \ln \varepsilon$, to depict the fractal dimension involved in this intermittent process, I represents the distribution of trajectory in the phase space and dimension is estimated by calculating the slope between I and box size ε. For the same noise intensity used in Fig. 1(b), we plotted the relation between I and ε in Fig. 1(c), the related information dimension is estimated to be $d \approx 0.34$. Together considering the largest Lyapunov exponent $\lambda \approx -0.05$, SNA is preliminary found. In order to investigate the relation between noise intensity D and fractal dimension d, we plot Fig. 1(d). A critical transition of dimension can be found at D_c. For $D < D_c$, d increases linearly as D increases, this is by no means than just the effect of noise, because during our estimation of the fractal dimension we employ the same range of box size ε, while the periodic attractor becomes thicker as D increases. But when $D \gtrsim D_c$, the fractal dimension has a distinct change in that it increases more rapidly (the scaling between D and d near D_c can be roughly estimated to be with a power law scaling and with the same exponent α as the relation between D and λ near D_c). The explanation of this transition as follows, for $D < D_c$, although occasionally there can be trajectory points be "kicked out" of the periodic attractor and stay a finite time in the chaotic saddle due to some sudden large noise forcing, this probability is small enough and doesn't effect the dimension in comparison with the increase of noise intensity. But for $D > D_c$, the dimension induced by unstable dimension variability becomes much strong than effect of the noise, and the fractal dimension now is dominated by the behavior of intermittency, more specifically, by temporal trajectories stay in the chaotic saddle. At the same time, as D increases, the frequency that a typical trajectory stays in the chaotic saddle also increases in a power law relation [13], as a result, d increases more rapidly after D_c and a transition of dimension at this point is formed. This dimension transition shows

us a new picture of the critical behavior near D_c, and may help our understanding of D_c from another different aspect,

In order to investigate the behavior of SNAs in more detail, except the two basic criteria of the nonpositive largest Lyapunov exponent and the fractal dimension, a host of other properties have also been used to characterize SNAs, such as properties of frequency spectrum and singular-continuous spectrum, properties of local Lyapunov exponent (LLE), and so on [2]. It is well known that the frequency spectrum of SNAs admits a power law relation $N(\sigma) \sim \sigma^{-\kappa}$ with $1 < \kappa < 2$, where the spectral distribution function $N(\sigma)$ is defined as the number of peaks in the Fourier power spectrum larger than some value σ. In Fig. 2(a) and (b) we plot the power spectrum and the relation between N and σ for trajectory of Fig. 1(b), respectively. The slope shown in Fig. 2(b) is estimated to be $\kappa \approx 0.86$. Another criterion for SNAs verifying is to analyze its singular-continuous spectrum, by calculating Fourier transform $X(\Omega, T) = \sum_{n=1}^{T} x_n e^{i2\pi n\Omega}$ with x_n the time series of trajectory and T the total time. For a proper frequency Ω, it was demonstrated that for SNAs the power scaling $|X(\Omega, T)|^2 \sim T^{\beta}$ is hold, and the exponent β is between 1 and 2 which represents a persistent motion (drift) and a random motion (Brownian motion), respectively. Also, the path for $(\mathrm{Re}\, X, \mathrm{Im}\, X)$ is expected to be fractal and self-similar [10, 15]. In our study, we use the same parameters as Fig. 1(b) and set $\Omega = (\sqrt{5} - 1)/2$, the golden number, plot $|X(\Omega, T)|^2$ versus T in Fig. 2(c) and $\mathrm{Re}\, X$ versus $\mathrm{Im}\, X$ in Fig. 2(d). The slope in Fig. 2(c) is estimated to be $\beta \approx 1.5$ and the path $(\mathrm{Re}\, X, \mathrm{Im}\, X)$ apparently exhibits a fractal and self-similar structure. These results again confirm the attractor in Fig. 1(b) is strange nonchaotic.

The largest Lyapunov exponent only give the asymptotic value which represents the average rate of separation of nearby trajectories, it can't give the temporal information and explore the internal dynamics. To overcome this shortcoming and also in order to detect the timely behavior of SNAs, local or finite-time Lyapunov exponent $\lambda_i(t)$ was defined and used for SNAs studies in former works [2, 11]. The definition of $\lambda_i(t)$ is similar to the largest Lyapunov exponent except that it is computed over a finite-time interval, t, the subscript i indexes the segment in which this exponent is evaluated. Depending on the difference of route for SNAs creation, the behavior of LLE is also different from each other, this property is also regarded as one of the important criteria for SNAs classification [2]. The utilization of LLE in our model is straightforward, when trajectory stays in the vicinity

of the periodic attractor, its behavior is similar to the original periodic state and LLE is negative, when trajectory stays in the vicinity of the chaotic saddle, on the contrary, the LLE will be positive. Another advantage for using LLE exists in its utilization in intermittency studies, because system is driven by noise, the return position of points escaping from chaotic saddle will locate to one of periodic states by random, this make it difficult to quantify the distance that trajectory leaves from the periodic attractor, but this problem doesn't exist for LLE in the sense that LLE only consider which sets, period or chaotic saddle, the local trajectory is stays, regardless which periodic state it locate. With $D = 10^{-5}$, in Fig. 3(a) we plot $\lambda_i(t)$ versus segment i for $t = 1000$, random bursting of LLE can be found and typical behavior of intermittency is shown. In order to explore its temporal and amplitude properties, we plot the laminar-phase and possibility distributions of λ_i in Fig. 3(b) and (c), respectively. Significant positive tail, as shown in Fig. 3(c), decays slowly as a function of λ_i, strongly indicates the distinction of intermittency and other routes of SNAs creation [11]. A direct consequence of this distinction is the variance of λ_i increases drastically after the transition point at D_c, as we plot in Fig. 3(d). These observations are also consistent with the intermittent trajectory (Fig. 1(b)) and the transition of dimension (Fig. 1(d)). The exponential law of laminar-phase in Fig. 3(c) indicates the happening of symmetry breaking near D_c [10, 13], similar to the type-III intermittency defined in Ref. [11].

We choose Duffing equation as our second model so as to investigate noise induced SNAs in continuous dynamical systems. As a typical nonautonomous system, Duffing equation owns a zero Lyapunov exponent which associated with the time axis, different to autonomous systems (like Lorenz, Rossler and models studied in Ref. [13]), the neutral dimension is always there regardless the noise intensity). Meanwhile, after noise intensity exceeds the critical value D_c, the nontrival largest Lyapunov exponent λ also increases with a power law scaling and changes to positive at D_0 [13], thus for Duffing model SNAs there also exists range $D_c < D < D_0$ where SNAs are expected (D_c and D_0 have the similar definitions as logistic map but for Duffing equation here). The functions we will study as follows,

$$\begin{aligned} \dot{x} &= y + D\xi, \\ \dot{y} &= -hy + (1 + A\cos t)x - x^3 + D\xi', \end{aligned} \tag{2}$$

with ξ and ξ' two independent Gaussian random variables of zero mean and unit variance, and D the noise intensity. For $h = 0.1$ and $A = 0.12$, we plot λ versus D in Fig. 4(a),

there two critical noise intensities, D_c and D_0, can be found and SNAs are expected within range $10^{-1.65} \approx D_c < D < D_0 \approx 10^{-1.3}$. When $D = 0$, the trajectory is located in periodic 4 window and $\lambda_p \approx -0.047$. For $D = 10^{-1.4}$, $\lambda \approx -0.03$, the time series of variable y is plotted in Fig. 4(b) and intermittency is shown. Fig. 4(c) shows the (x, y) projection of a trajectory of 10000 iterations (after 5000 preiterations) on the stroboscopic surface of section defined by $t' = 2n\pi(n = 1, 2, ...)$ for the same parameter of Fig. 4(b), geometric shape of the attractor appears to be strange and related information dimension is estimated to be $d \approx 1.3$. In Fig. 4(d) we plot the relation between noise intensity D and information dimension d, again, we can find a transition near D_c and d increases rapidly once noise intensity exceeds this critical value. For further study of other properties associated with SNAs, in Fig. 5(a) we plot the singular-continuous spectrum, the subplot in Fig. 5(a) is the path of $(\mathrm{Re}\, X, \mathrm{Im}\, X)$ with X the power amplitude of Fourier transform, the slope, $\beta \approx 1.64$, and fractal structure again confirm the existence of SNA. In order to investigate its intermittent behavior in detail, we plot local Lyapunov exponent $\lambda_i(t)$ versus i with $t = 200$ in Fig. 5(b), laminar-phase in Fig. 5(b) and possibility distributions in Fig.5 (d), respectively. All these results indicate that this is typically an intermittent behavior and system undergoes a kind of symmetry breaking. In our simulations, we also test the variance of λ_i for Duffing model, just as we find for logistic map, a sudden transition appears near D_c also.

Since the situation where two coexisting dynamical invariant sets with distinct unstable dimensions can linked by noise can occur in any periodic window and states where periodic attractor and isolated saddle periodic orbits coexist, and also because noise is ubiquitous in real systems, the phenomenon of SNAs induced by noise is expected to be fairly common and general. At the same time, SNAs not only can be found in transitions from period to chaos, but also can be found in transitions from chaos to period and in noisy synchronization systems. For example, for the model used in Ref. [16] with noise level $\sigma^2 = 0.3$ (details about function and parameters please refer to the same reference), the largest Lyapunov exponent $\lambda \approx -0.3$, while the related information dimension is estimated with $d \approx 1.55$, a behavior of SNA. Moreover, the above argument can be extended to higher-dimensional systems directly [13]. As we have mentioned in the first paragraph, its still remains an open question to construct a SNA in any chaotic system, by this study we find that the mechanism of noise induced SNAs is model independent and universal, so we think this is a more generic method for SNA creation and can be utilized both in theory and experiment. Meanwhile,

this kind of SNAs are not only typical, but also robust as well [17], since the system itself is under the perturbation of noise. We have also tried perturbations in other variables and parameters in models studied in this paper, the existence of SNAs is indeed robust within a wide range. To conclude, we summarize the common features for noise induced SNAs: (i) It is expected both in map system and nonautonomous continuous system. For autonomous continuous system the neutral dimension will be broken and the zero Lyapunov exponent changes to positive after D_c, thus SNAs maybe can't be found. (ii) This is typically an intermittent behavior and associates with some kind of symmetry breaking. (iii) Both the dimension and the variance of local Lyapunov exponent have a transition at D_c, which reflect the change of dynamical structure from other points of view. (iv) The mechanism is general and the phenomenon is robust.

We wish this work can help our comprehension in studies like transitions to chaos, noise induced synchronization [16], particle floating on a moving fluid [6], chaos control in noisy systems [18], and so on. Also, we wish the phenomenon of noise induced SNAs can be observed in experiments in the near future.

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Captions of Figures

Fig. 1 For noise driven logistic map. (a) The largest Lyapunov exponent λ versus noise intensity D, SNA is expected within range $10^{-5.08} \approx D_c < D < D_0 \approx 10^{-4.95}$. (b) $D = 10^{-5.0}$, $\lambda \approx -0.05$, intermittent behavior of trajectory, and (c), the related fractal information dimension $d \approx 0.34$. (d) Fractal dimension d versus D, a transition exists near D_c.

Fig. 2 Analysis of trajectory shown in Fig. 1(b). (a) Power spectrum amplitude A versus frequency f, and (b) the related spectrum distribution with slope $\kappa \approx 0.8$. (c) Singular-continuous spectrum $|X|^2$ versus time with slop $\beta \approx 1.5$, and (d) fractal and self-similar path of $(\mathrm{Re}(X), \mathrm{Im}(X))$.

Fig. 3 Parameters same to Fig. 1(b). (a) Intermittency behavior of local Lypunov exponent, (b) the power law laminar-phase of (a). (c) Possibility distribution of λ_i shown in (a), and (d) variance of λ_i versus noise.

Fig. 4 For noise driven Duffing equation. (a) The largest Lyapunov exponent λ versus noise intensity D, SNA is expected within range $10^{-1.65} \approx D_c < D < D_0 \approx 10^{-1.3}$. (b) $D = 10^{-1.4}$, $\lambda \approx -0.047$, the intermittency of variable y, and (c) projection of trajectory on the stroboscopic surface section, the related fractal information dimension $d \approx 1.3$. (d) Fractal dimension d versus D, a transition exists near D_c.

Fig. 5 Parameters same to Fig. 4(b). (a) singular-continuous spectrum $|X|^2$ versus time with slope $\beta \approx 1.64$. (b) Intermittency behavior of local Lypunov exponent, and (c) the power law laminar-phase of (b). (d) Possibility distribution of λ_i shown in (a).

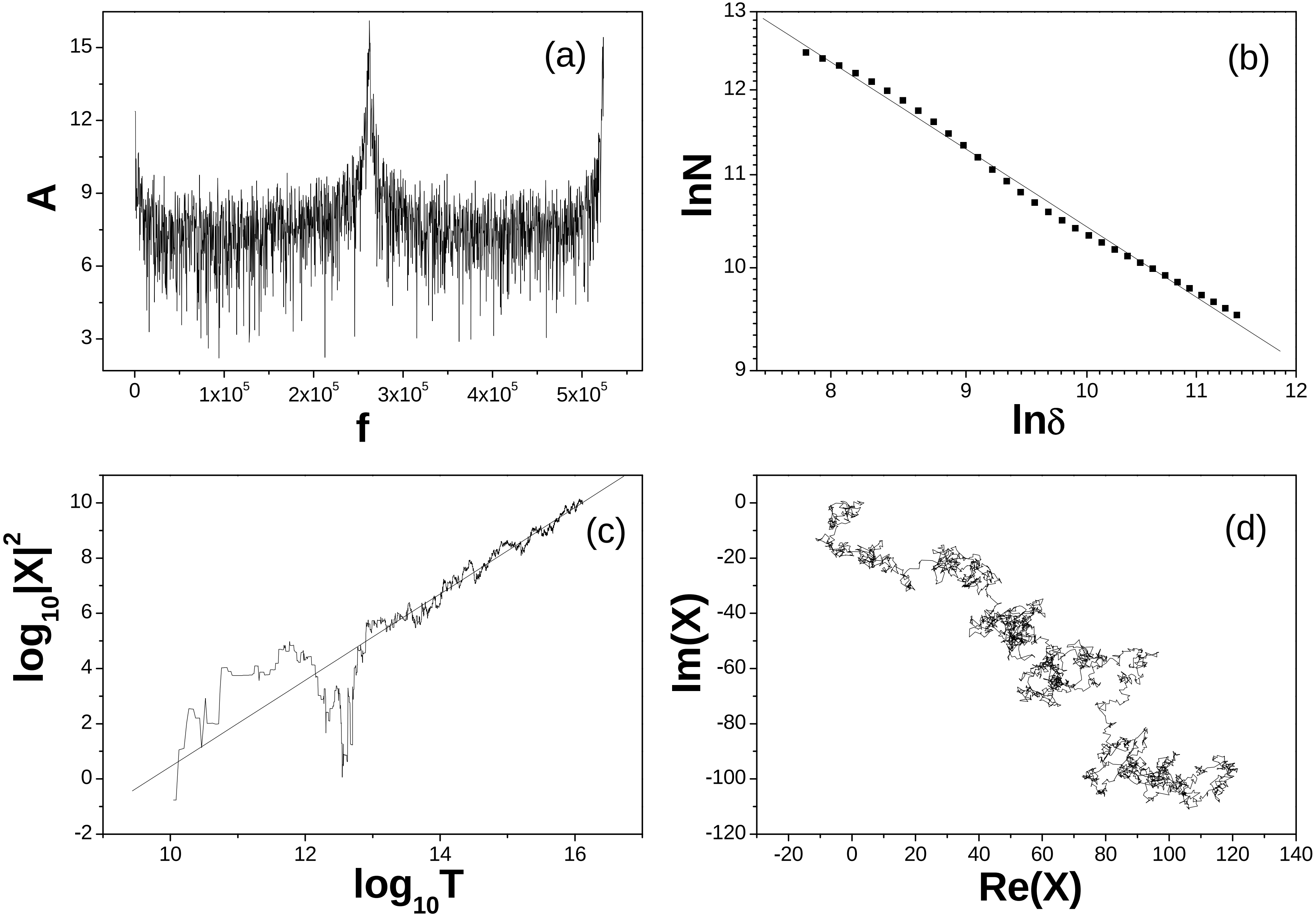

(a)
A
f
0
1x10^5
2x10^5
3x10^5
4x10^5
5x10^5
3
6
9
12
15
(b)
lnN
lnδ
9
10
11
12
13
8
9
10
11
12
(c)
log10|X|^2
log10T
-2
0
2
4
6
8
10
10
12
14
16
(d)
Im(X)
Re(X)
0
-20
-40
-60
-80
-100
-120
-20
0
20
40
60
80
100
120
140

Fig. 2

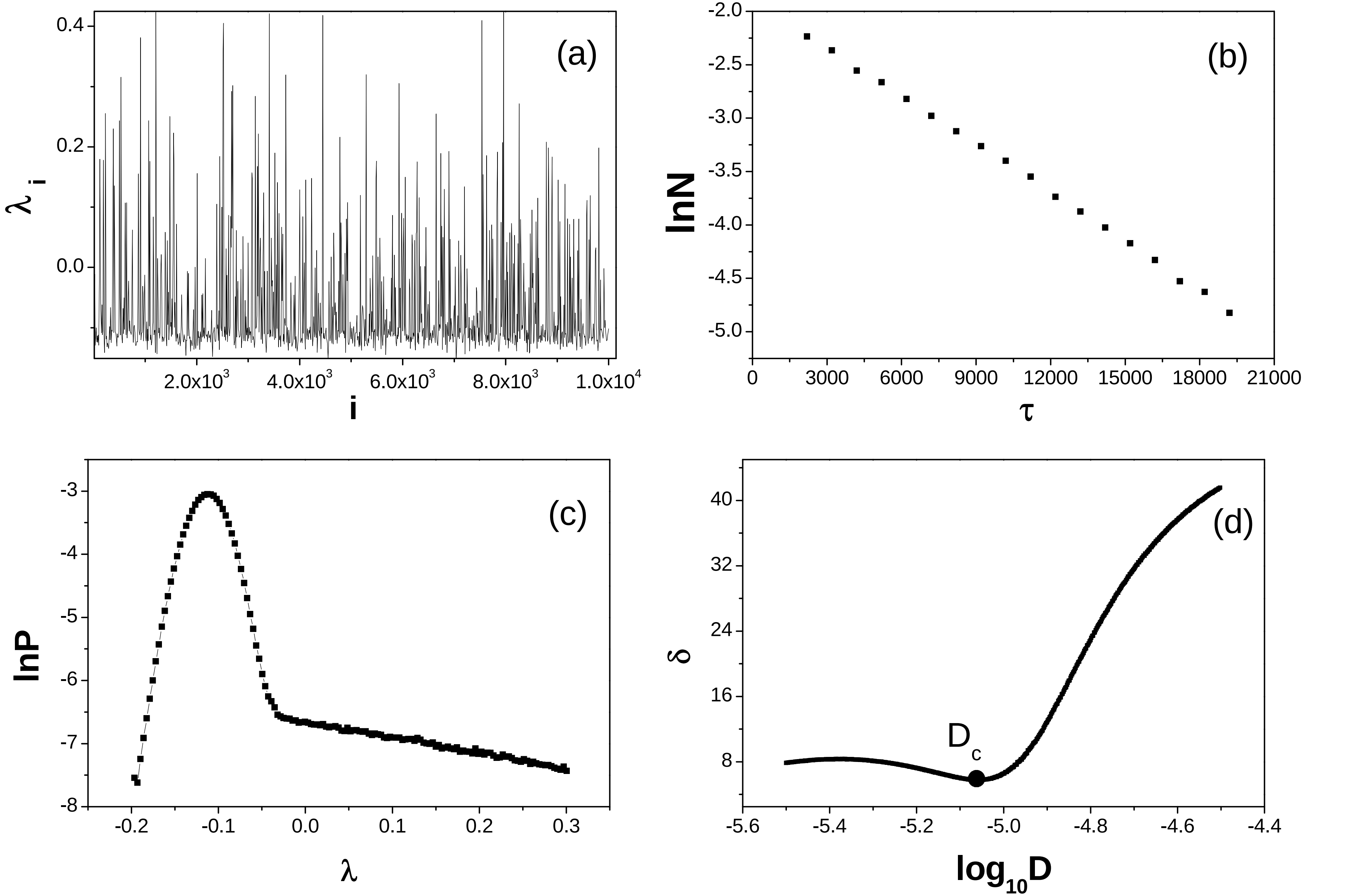
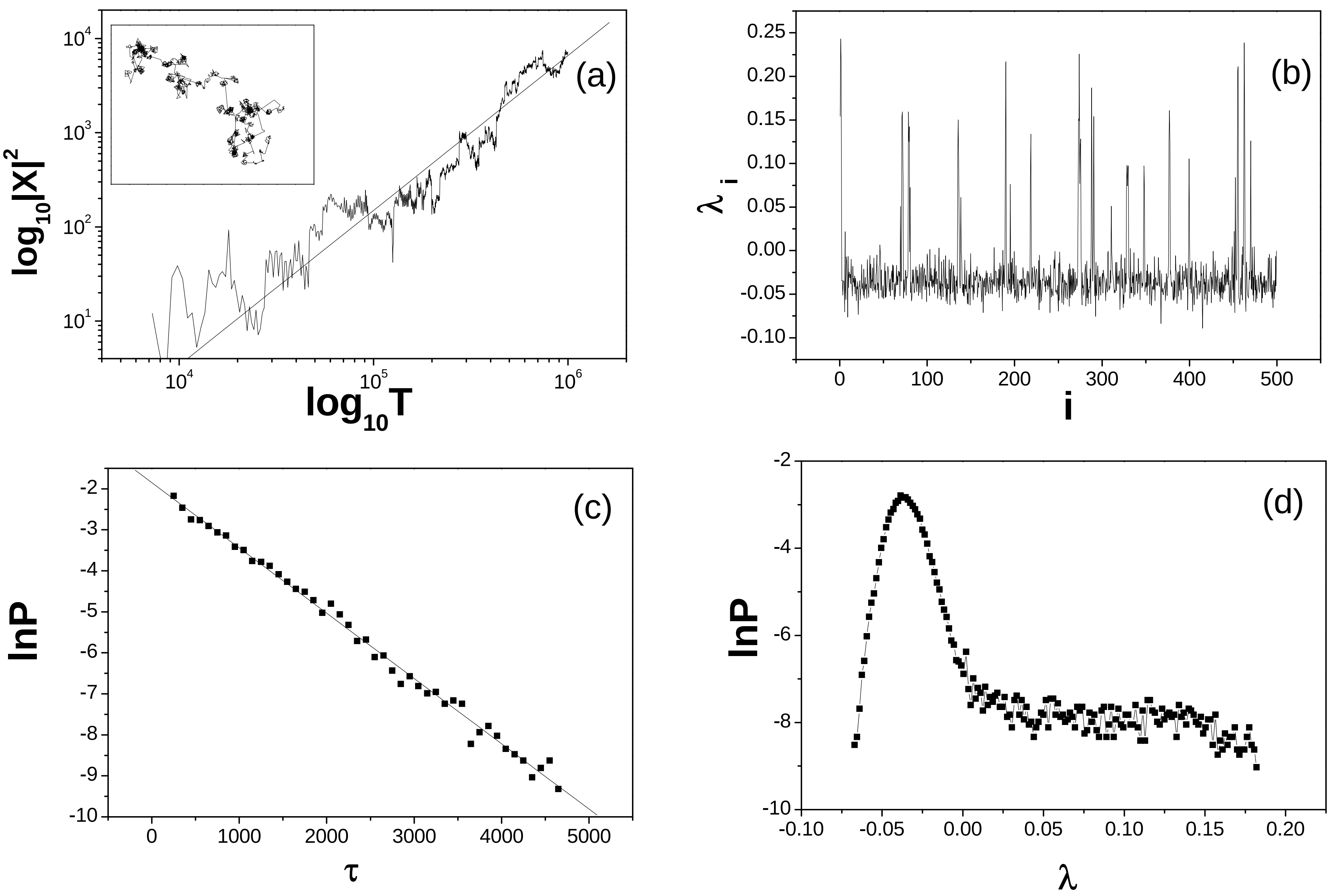

(a)
λ_i
i
0.4
0.2
0.0
2.0x10^3
4.0x10^3
6.0x10^3
8.0x10^3
1.0x10^4
(b)
lnN
τ
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
-5.0
0
3000
6000
9000
12000
15000
18000
21000
(c)
lnP
λ
-3
-4
-5
-6
-7
-8
-0.2
-0.1
0.0
0.1
0.2
0.3
(d)
δ
$\log_{10}D$
D_c
40
32
24
16
8
-5.6
-5.4
-5.2
-5.0
-4.8
-4.6
-4.4

Fig. 3



(a)
$\log_{10}|X|^2$
$\log_{10}T$
10^1
10^2
10^3
10^4
10^4
10^5
10^6
(b)
λ_i
i
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
-0.10
0
100
200
300
400
500
(c)
lnP
τ
-2
-3
-4
-5
-6
-7
-8
-9
-10
0
1000
2000
3000
4000
5000
(d)
lnP
λ
-2
-4
-6
-8
-10
-0.10
-0.05
0.00
0.05
0.10
0.15
0.20

Fig.5